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Symmetry reductions of partial differential equations related to singular manifold expansions

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Abstract. It is shown that all the information yielded by the reduction methods of Bluman and Cole and Clarkson and Kruskal can be obtained using the singular manifold expansion. The Burgers' equation, modified Korteweg–de Vries equation, Caudrey–Dodd–Gibbon equation and the Fitzhugh–Nagumo equation are used as illustrative examples. Several new exact solutions are presented.

1. Introduction

Recently, there has been considerable interest in symmetry reductions of partial differential equations (PDEs), mainly because the procedure reduces the number of independent variables, and, therefore, assists in the determination of exact solutions. The classical method [1–4] for this purpose is to use the symmetry properties of the PDE: any group of point symmetries admitted by the equation defines a symmetry reduction. This method has been used to study Burgers' equation by Tajiri *et al* [5], the modified Korteweg–de Vries equation by Lakshmanan and Kaliappan [6] and the Fitzhugh–Nagumo equation by Nucci and Clarkson [7].

A generalization of the above method called the 'non-classical method' was proposed by Bluman and Cole [8]. The family of solutions is now larger than that obtained with the classical method. This technique has been further generalized by Olver and Rosenau [9].

A direct method which does not use group analysis techniques for determining the symmetry reductions of a given PDE

$$Lu = 0 \tag{1.1}$$

for a function of two variables $u(x,t)$ (where L is a differential operator) has been developed by Clarkson and Kruskal [10]. In particular, they obtained solutions in the form

$$u(x,t) = A(x,t)F(z) + B(x,t) \tag{1.2}$$

with $A(x,t)$, $F(z)$ and $B(x,t)$ as functions of their arguments, and z as a similarity variable. Their reduction procedure suggested that

$$z(x,t) = \theta(t)x + \sigma(t) \tag{1.3}$$

with θ and σ as arbitrary functions of t. Their method proved effective in framing new symmetry solutions to a variety of PDEs. Later, Levi and Winternitz [11] and Pucci and Succomandi [12] recognized that solutions derived by the direct method of Clarkson and

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Kruskal are always invariant solutions under non-classical symmetries admitted by the equation. However, PDEs may admit symmetry reductions with non-classical symmetries, yet they are not recoverable by the direct method [7, 13]. A consistency criterion has been formulated by Arrigo *et al* [14] which, if satisfied, ensures that the methods of Bluman and Cole and Clarkson and Kruskal yield the same results.

The singular manifold method [15, 16] has so far been used to obtain travelling-wave-type non-classical solutions [17–19]. The solution of PDE (1.1) is now assumed to be in the series form

$$u(x,t)=\sum_{k=0}^{\infty}u_k(x,t)z^{k+\alpha} \tag{1.4}$$

and the expansion function z is specified in such a way that the *consistency conditions* are satisfied on the singular manifold itself. This, in turn, corresponds to the truncation of series and contains certain symmetry information. However, if z plays the role of a similarity variable, the infinite series solution (1.4) may yield a more general travelling-wave-type non-classical symmetry solution. The expansions truncated at the constant level term have also been employed to construct the Bäcklund transformation and Lax pairs for various PDEs (see [20–23]).

In this paper we use the singular manifold expansion (1.4) to discuss the symmetry reductions of Burgers' equation, the modified Korteweg–de Vries equation, the Caudrey–Dodd–Gibbon equation and the Fitzhugh–Nagumo equation. Specifically, we show how these expansions can be systematically used to recover all those symmetry solutions which could be obtained through the direct method of Clarkson and Kruskal; the truncated expansion renders the solutions found by using Bluman and Cole's non-classical approach. In fact, this method opens up many new possibilities in a natural way.

2. Symmetry reductions of Burgers' equation

In this section we seek reductions of the Burgers' equation

$$u_t+uu_x+u_{xx}=0 \tag{2.1}$$

with the help of series (1.4). When this series solution is substituted into (2.1), the leading order analysis shows that

$$u_0=2z_x \tag{2.2}$$

and $\alpha=-1$. Here, we may note that if u_0 is a function of t alone, z in (2.2) can be defined as in (1.3). If z is an arbitrary expansion function defining the singularity manifold by $z=0$, the infinite expansion can be truncated, provided z satisfies a nonlinear PDE, as we shall see later. Thus, there are two freedoms in the determination of z.

Case 1. $z_{xx}=0$. We introduce z as defined in (1.3) and t as the new independent variables in (2.1) to get

$$u_t+u_z(Rz+S+\theta u)+\theta^2u_{zz}=0 \tag{2.3}$$

where

$$R=\frac{\mathrm{d}\theta}{\mathrm{d}t}\theta^{-1}\qquad S=\frac{\mathrm{d}\sigma}{\mathrm{d}t}-\sigma R \tag{2.4}$$

with θ and σ as arbitrary functions of t. We may now re-write the series solution (1.4) as

$$u(z,t)=A(z,t)F(z)+B(z,t) \tag{2.5}$$

where

$$A(z, t) = A_0 + A_1 z + A_2 z^2 + \cdots \tag{2.6}$$

$$B(z, t) = B_0 + B_1 z + B_2 z^2 + \cdots \tag{2.7}$$

and

$$F(z) = z^{-1}(F_0 + F_1 z + F_2 z^2 + \cdots) \tag{2.8}$$

with A_k, B_k, $k \geqslant 0$, as arbitrary functions of t, and F_k as arbitrary constants. In writing these expansions, A and B are assumed to have fixed singularities in the z-plane. The movable pole $z = 0$ does not result from these functions.

We substitute (2.5) with (2.6)–(2.8) into (2.3) and equate the coefficients of different powers of z to zero. This leads to

$$z^{-3} : 2A_0F_0\theta^2 - \theta A_0^2 F_0^2 = 0 \tag{2.9}$$

$$z^{-2} : -A_0F_0(S + \theta(A_0F_1 + A_1F_0 + B_0)) = 0 \tag{2.10}$$

$$z^{-1} : -RA_0F_0 + \frac{\mathrm{d}A_0}{\mathrm{d}t}F_0 = 0 \tag{2.11}$$

$$\begin{aligned} z^0 : {} & (A_0F_2 + A_1F_1 + A_2F_0 + B_1)(S + \theta(A_0F_1 + A_1F_0 + B_0)) \\ & + (A_0F_3 + A_1F_2 + A_2F_1 + A_3F_0 + B_2)(\theta A_0F_0 + 2\theta^2) \\ & + \frac{\mathrm{d}A_1}{\mathrm{d}t}F_0 + \frac{\mathrm{d}A_0}{\mathrm{d}t}F_1 + \frac{\mathrm{d}B_0}{\mathrm{d}t} = 0. \end{aligned} \tag{2.12}$$

Equations (2.9)–(2.12) yield

$$A_0 = \theta \tag{2.13}$$

$$A_1 = \theta^{-2}C_2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{S}{\theta}\right) + C_3\theta - B_2/F_2 \tag{2.14}$$

$$B_0 = -(A_1F_0 + C_1\theta + S/\theta) \tag{2.15}$$

where

$$F_0 = 2 \qquad C_1 = F_1 \qquad C_2 = (4F_2)^{-1} \qquad C_3 = F_3/F_2. \tag{2.16}$$

The solution (2.5) may now be read as

$$u(z, t) = (\theta + A_1 z)F(z) - (A_1F_0 + C_1\theta + s/\theta) + B_1 z + B_2 z^2 \tag{2.17}$$

where B_1 and B_2 are arbitrary functions of t, and A_1 is given by (2.14). Substitution of (2.17) into (2.3) gives

$$\begin{aligned} & [\theta^3 F + 2\theta^2 A_1 zF - C_1\theta^3 + [(R + B_1\theta)\theta - A_1(2A_1\theta + c_1\theta^2)]z]\frac{\mathrm{d}F}{\mathrm{d}z} \\ & + [\theta A_1^2 z^2 F + [\theta^2 B_2 + A_1(R + B_1\theta)]z^2 + \theta A_1 B_2 z^3]\frac{\mathrm{d}F}{\mathrm{d}z} \\ & + \theta^2 A_1 F^2 + \theta A_1^2 zF^2 + \left[-A_1(2A_1\theta + c_1\theta^2) + B_1\theta^2 + \frac{\mathrm{d}\theta}{\mathrm{d}t}\right]F + 3\theta A_1 B_2 z^2 F \\ & + \left[A_1(R + B_1\theta) + \theta A_1 B_1 + 2\theta^2 B_2 + \frac{\mathrm{d}A_1}{\mathrm{d}t}\right]zF - B_1(2A_1\theta + C_1\theta^2) \\ & + \frac{\mathrm{d}B_0}{\mathrm{d}t} + 2B_2\theta^2 + \left[B_1(R + B_1\theta) - 2B_2(2A_1\theta + C_1\theta^2) + \frac{\mathrm{d}B_1}{\mathrm{d}t}\right]z \\ & + \left[2B_2(R + B_1\theta) + \theta B_1 B_2 + \frac{\mathrm{d}B_2}{\mathrm{d}t}\right]z^2 + 2\theta B_2^2 z^3 + \theta^2(\theta + A_1 z)\frac{\mathrm{d}^2F}{\mathrm{d}z^2} = 0. \end{aligned} \tag{2.18}$$

It can be readily noted that if (2.18) is to become an ordinary differential equation for $F(z)$, we must have

$$A_1 = C_4\theta \qquad B_1 = C_5\theta - \frac{d\theta}{dt}\theta^{-2} \qquad B_2 = C_6\theta. \tag{2.19}$$

Since the contribution of those terms which are proportional to θ is insignificant, we can set

$$C_1 = 0 \qquad C_2 = 0 \qquad C_4 = 0 \qquad C_5 = 0 \qquad \text{and } C_6 = 0.$$

Further, if θ and σ are the solutions of the following ordinary differential equations:

$$\theta\frac{d^2\theta}{dt^2} - 2\left(\frac{d\theta}{dt}\right)^2 = -C_7\theta^6 \tag{2.20}$$

$$\theta\frac{d^2\sigma}{dt^2} - 2\frac{d\theta}{dt}\frac{d\sigma}{dt} = (C_3 - C_7\sigma)\theta^5 \tag{2.21}$$

with C_3 and C_7 as an arbitrary constant, the solution (2.17) may be re-written as

$$u = \theta F(z) - \theta^{-1}\left(\frac{d\theta}{dt}x + \frac{d\sigma}{dt}\right). \tag{2.22}$$

Here, F satisfies

$$\frac{d^2F}{dz^2} + F\frac{dF}{dz} + C_7z - C_3 = 0. \tag{2.23}$$

This symmetry reduction has been obtained by Clarkson and Kruskal using their direct method. Equation (2.23) can be integrated once to give

$$\frac{dF}{dz} + (F^2 + C_7z^2)/2 - C_3z = C_8 \tag{2.24}$$

where C_8 is the constant of integration. Setting $F = 2(d\psi/dz)\psi^{-1}$ in (2.24) yields

$$\frac{d^2\psi}{dz^2} + (C_7z^2 - C_3z - C_8)\frac{\psi}{2} = 0. \tag{2.25}$$

This equation has many solutions for $C_3 = 0$ in addition to the special case when $C_3 \neq 0$.

If we set $C_3 = 0$, $C_7 = -1$, $C_8 = -(2\nu + 1)$ and $y = 2^{-1/2}z$, (2.25) takes the form

$$\frac{d^2\psi}{dy^2} + (2\nu + 1 - y^2)\psi = 0 \tag{2.26}$$

which is the parabolic cylinder equation with linearly independent solutions $D_\nu(y)$ and $D_\nu(-y)$; ν is a parameter. In the special case when $\nu = n$ is a positive integer, we have

$$D_n(y) = \mathrm{He}_n(y)\exp[-y^2/2] \tag{2.27}$$

where $\mathrm{He}_n(y)$ is the Hermite polynomial given by

$$\mathrm{He}_n(y) = (-1)^n \exp[y^2]\frac{d^n}{dy^n}[\exp(-y^2)]. \tag{2.28}$$

If $C_3 = 2$, $C_7 = 0$ and $C_8 = 0$, (2.25) becomes an Airy equation and its general solution can be written as

$$\psi = b_1(-z)^{1/2}I_{1/3}(2(-z)^{3/2}/3) + b_2(-z)^{1/2}K_{1/3}(2(-z)^{3/2}/3) \tag{2.29}$$

where $I_{1/3}(2(-z)^{3/2}/3)$ and $K_{1/3}(2(-z)^{3/2}/3)$ are modified Bessel functions of first and second kind of order 1/3, respectively.

If $C_3 = 0$, $C_7 = 0$ and $C_8 \neq 0$, the general solution of (2.25) is given by

$$\psi = b_1 \exp[(C_8/2)^{1/2} z] + b_2 \exp[-(C_8/2)^{1/2} z]. \tag{2.30}$$

Here, b_1 and b_2 are arbitrary constants.

In the next part of this section we shall see that (2.24) has many exact solutions if $C_3 = 0$ and $C_7 = -1/4$.

Case 2. $z_{xx} \neq 0$. We now seek the symmetry reductions of Burgers' equation (2.1) by using the truncated expansion

$$u(x,t) = \frac{u_0(x,t)}{z} + u_1(x,t) \tag{2.31}$$

with u_0, u_1 and z as arbitrary functions. When this solution is substituted into the PDE and the coefficients of different powers of z are set equal to zero, we obtain

$$z^{-3} : -u_0^2 z_x + 2u_0 z_x^2 = 0 \tag{2.32}$$

$$z^{-2} : -u_0 z_t + u_0 u_{0x} - u_0 u_1 z_x - 2u_{0x} z_x - u_0 z_{xx} = 0 \tag{2.33}$$

$$z^{-1} : u_{0t} + u_0 u_{1x} + u_1 u_{0x} + u_{0xx} = 0 \tag{2.34}$$

$$z^0 : u_{1t} + u_1 u_{1x} + u_{1xx} = 0. \tag{2.35}$$

We have observed that if the term $u_2 z$ is present in the solution (2.31), both u_{1x} and z_x must be functions of t only. This implies that z is in the form (1.3).

For now equation (2.32) gives u_0 as in (2.2). On using this in (2.33), we get

$$z_t + u_1 z_x + z_{xx} = 0. \tag{2.36}$$

Here, equation (2.34) is satisfied. If u_1 is an arbitrary constant, say c_*, (2.35) is also exactly satisfied. We now consider the following cases:

(I) If $c_* = 0$, we have the solution

$$u = 2z_x/z \tag{2.37}$$

where z satisfies the heat equation

$$z_t + z_{xx} = 0. \tag{2.38}$$

Thus, we have recovered the well known result that the Cole–Hopf transformation maps the Burgers' equation into the linear heat equation (see also [15]).

We summarize the solutions of type (2.37) as follows.

(a) PDE (2.38) has a solution of the form

$$z(x,t) = \sum_{k=0}^{n} \sum_{i=1}^{2(k+1)} b_{k(k+1)+i} x^{i-1} t^{n-k} \tag{2.39}$$

with b_i, $i \geqslant 1$ as arbitrary constants to be determined, and n as an arbitrary integer.

If we set $n = 2$, the solution of (2.38) may be written as

$$z(x,t) = b_1 + b_2 x + b_3 x^2 + b_4 x^3 + b_5 x^4 + b_6 x^5 - (2b_3 + 6b_4 x + 12b_5 x^2 + 20b_6 x^3)t \\ +(12b_5 + 60b_6 x)t^2. \tag{2.40}$$

For $b_1 = -a_2$, $b_2 = 2a_1$, $b_3 = 1$, $b_4 = 0$, $b_5 = 0$, $b_6 = 0$, we obtain the solution (iii$_a$) of Pucci (or the solution (4.6) of Arrigo *et al* [14]). This solution can be obtained by both the Bluman and Cole method and the direct method of Clarkson and Kruskal, for the consistency criterion of Arrigo *et al* is satisfied. Nevertheless, we have found that if the constants b_1, b_2, b_3, b_4, b_5 and b_6 in (2.40) are chosen such that the expression $b_1+b_2x+b_3x^2+b_4x^3+b_5x^4+b_6x^5$ is in the form $a_1(x+a_2)^p$, $p \leqslant 5$, the solution (2.37) with

(2.40) is always recoverable by the direct method. For these solutions we have $\theta(t) = t^{-1/2}$, $\sigma(t) = a\theta$ in (1.3), and $C_3 = 0$; $C_7 = -\frac{1}{4}$; $C_8 = -5/2, -7/2, -9/2, -11/2, \ldots$ in (2.24). For $b_1 = -2b$, $b_2 = -2a$, $b_3 = 0$, $b_4 = -1$, $b_5 = 0$, $b_6 = 0$, we have the solution in the form (4.27) of Arrigo *et al* . This solution cannot be obtained by the direct method.

Setting $b_2 = 0$, $b_4 = 1$, $b_5 = 0$, $b_6 = 0$, we obtain a new two-parameter family of solutions:

$$u(x,t) = \frac{4b_3x + 6x^2 - 12t}{b_1 + b_3x^2 + x^3 - (2b_3 + 6x)t}. \tag{2.41}$$

Setting $b_5 = 1$, $b_6 = 0$ and $b_1 \neq (b_4/4)^4$, we obtain a new four-parameter family of solutions:

$$u(x,t) = [2b_2 + 4b_3x + 6b_4x^2 + 8x^3 - (12b_4 + 48x)t]/D_1 \tag{2.42}$$

where

$$D_1 = b_1 + b_2x + b_3x^2 + b_4x^3 + x^4 - (2b_3 + 6b_4x + 12x^2)t + 12t^2. \tag{2.43}$$

Setting $b_6 = 1$, $b_1 \neq (b_5/5)^5$, we obtain a new five-parameter family of solutions:

$$u(x,t) = [2b_2 + 4b_3x + 6b_4x^2 + 8b_5x^3 + 10x^4 - (12b_4 + 48b_5x + 120x^2)t + 120t^2]/D_2 \tag{2.44}$$

where

$$D_2 = b_1 + b_2x + b_3x^2 + b_4x^3 + b_5x^4 + x^5 - (2b_3 + 6b_4x + 12b_5x^2 + 20x^3)t + (12b_5 + 60x)t^2. \tag{2.45}$$

As a matter of fact, one can produce many new solutions by just fixing a different n in (2.39).

(b) Since the PDE (2.38) is linear, and

$$z = b_7 \exp[b_8x - b_8^2t] \tag{2.46}$$

is also its solution, the superposition principle holds. This gives a seven-parameter family of solutions:

$$u(x,t) = [2b_2 + 4b_3x + 6b_4x^2 + 8b_5x^3 + 10b_6x^4 - (12b_4 + 48b_5x + 120b_6x^2)t + 120b_6t^2 + 2b_7b_8 \exp[b_8x - b_8^2t]]/D_3 \tag{2.47}$$

where

$$D_3 = b_1 + b_2x + b_3x^2 + b_4x^3 + b_5x^4 + b_6x^5 - (2b_3 + 6b_4x + 12b_5x^2 + 20b_6x^3)t + (12b_5 + 60x)t^2 + b_7 \exp[b_8x - b_8^2t]. \tag{2.48}$$

For $b_1 = -c_1$, $b_2 = 1$, $b_3 = 0$, $b_4 = 0$, $b_5 = 0$, $b_6 = 0$ in (2.47)–(2.48), and $b_7 = \exp[-c_2g^2]$ and $b_8 = g$, we obtain the solution (iii$_c$) of Pucci.

(II) If $c_* \neq 0$, the solution of (2.1) is given by

$$u(x,t) = (2z_x + c_*z)/z \tag{2.49}$$

where z is the solution of the PDE

$$z_t + c_*z_x + z_{xx} = 0. \tag{2.50}$$

We now have the following solutions of Burgers' equation:

(c) The solution of (2.50) may now be assumed in the form

$$z(x,t)=\sum_{k=0}^{n}\sum_{i=0}^{k} b_{i+1+k(k+1)/2}x^i t^{n-k} \tag{2.51}$$

with b_i, $i \geqslant 1$, as the arbitrary constants to be determined and n as an arbitrary integer. For $n=3$, we have (2.51) as

$$z(x,t)=\frac{-c_*b_1}{3}t^3+\left(-\frac{c_*b_2}{2}+\frac{b_1}{c_*}+b_1x\right)t^2+\left(-c_*b_3+\frac{1}{c_*}\left(b_2+\frac{2b_1}{c_*^2}\right)+b_2x-\frac{b_1}{c_*}x^2\right)t$$
$$+b_4+b_3x-\frac{1}{2c_*}\left(b_2+\frac{2b_1}{c_*^2}\right)x^2+\frac{b_1}{3c_*^2}x^3. \tag{2.52}$$

Furthermore, since the solution

$$z(x,t)=(b_5+b_6x)\exp\left(\frac{-c_*x}{2}+\frac{c_*t^2}{4}\right)+b_7\exp(-c_*x) \tag{2.53}$$

also satisfies (2.50), the superposition principle holds, and we may obtain a seven-parameter family of solutions:

$$u(x,t)=\left[2b_3+c_*b_4+\left(-\frac{4b_1}{c_*^3}-\frac{2b_2}{c_*}+c_*b_3\right)x+\left(\frac{b_1}{c_*^2}-\frac{b_2}{2}\right)x^2+\frac{b_1}{3c_*}x^3\right.$$
$$+\left\{\frac{2b_1}{c_*^2}+3b_2-c_*^2b_3+\left(-\frac{4b_1}{c_*}+c_*b_2\right)x-b_1x^2\right\}t$$
$$+\left(3b_1-\frac{c_*^2b_2}{2}+c_*b_1x\right)t^2-\frac{c_*^2b_1}{3}t^3$$
$$\left.+2b_6\exp\left[-\frac{c_*x}{2}+\frac{c_*t^2}{4}\right]-c_*b_7\exp[-c_*x]\right]$$
$$\times\left[-\frac{c_*b_1}{3}t^3+\left(-\frac{c_*b_2}{2}+\frac{b_1}{c_*}+b_1x\right)t^2\right.$$
$$+\left\{-c_*b_3+\frac{1}{c_*}\left(b_2+\frac{2b_1}{c_*^2}\right)+b_2x-\frac{b_1}{c_*}x^2\right\}t+b_4+b_3x$$
$$-\frac{1}{2c_*}\left(b_2+\frac{2b_1}{c_*^2}\right)x^2+\frac{b_1}{3c_*^2}x^3+(b_5+b_6x)\exp\left[-\frac{c_*x}{2}+\frac{c_*t^2}{4}\right]$$
$$\left.+b_7\exp[-c_*x]\right]^{-1}. \tag{2.54}$$

For $b_1=b_2=b_3=b_4=0$, and $b_5=\pm 1$, $b_6=0$, $c_*=-f$, $b_7=b_{20}b_8$, $b_8=b_{10}^{1/2}$, we obtain solution (4.21) of Pucci (or the form (4.18) of Arrigo *et al*).

3. Symmetry reductions of the modified Korteweg–de Vries equation

In this section we use the Laurent series (1.4) to determine the symmetry solutions of modified Korteweg–de Vries equation

$$u_t+au^2u_x+bu_{xxx}=0. \tag{3.1}$$

Here, we have either $a=-3$, $b=2\delta^2$, or $a=1$, $b=1$, or $a=-3\delta/2$, $b=\delta$ (see [10, 16, 21]); δ is an arbitrary constant. For now $\alpha=-1$, and

$$u_0=I_0z_x \tag{3.2}$$

with $I_0 = \pm(-6b/a)^{1/2}$. Again, we may either consider z in the form (1.3) or truncate the series (1.4) to obtain a more general functional form of z.

Case 1. $z_{xx} = 0$. Introduction of z as in (1.3) and t as independent variables in (3.1) leads to

$$u_t + (Rz + S + a\theta u^2)u_z + b\theta^3 u_{zzz} = 0 \tag{3.3}$$

where R and S are defined as in (2.4), with θ and σ as arbitrary functions of t. When the solution of (3.3) is assumed to be in the series form (2.5), with A, B and F as given by (2.6), (2.7) and (2.8), respectively, we get the following algebraic relations for A_k, B_k and F_k, $k \geqslant 0$:

$$z^{-4} : -(a\theta A_0^3 F_0^3 + 6b\theta^3 A F_0) = 0 \tag{3.4}$$

$$z^{-3} : -2a\theta A_0^2 F_0^2 (A_0 F_1 + A_1 F_0 + B_0) = 0 \tag{3.5}$$

$$z^{-2} : -A_0 F_0 [S + a\theta((A_0 F_1 + A_1 F_0 + B_0)^2 + 2A_0 F_0 (A_0 F_2 + A_1 F_1 + A_2 F_0 + B_1))] = 0 \tag{3.6}$$

$$z^{-1} : F_0 \frac{\mathrm{d}A_0}{\mathrm{d}t} - A_0 F_0 R - 2a\theta A_0 F_0 (A_0 F_1 + A_1 F_0 + B_0)(A_0 F_2 + A_1 F_1 + A_2 F_0 + B_1) = 0. \tag{3.7}$$

Equation (3.4) gives

$$A_0 = \theta \tag{3.8}$$

and $F_0 = I_0$, with I_0 as in (3.2). Since F_k, $k \geqslant 0$, are constants, the functions A_1, A_2, A_3, B_0, B_1 and B_2 must be proportional to θ. Henceforth we set $A_{k+1} = 0$, $B_k = 0$, $k \geqslant 0$. In addition, $F_1 = 0$ and $F_k = 0$, $k \geqslant 3$; F_2 is an arbitrary constant. Equation (3.6) now yields

$$S = C_1 \theta^3 \tag{3.9}$$

where $C_1 = -aI_0 F_2$. Thus, the symmetry reduction of (3.2) has the form

$$u(z, t) = \theta(t) F(z). \tag{3.10}$$

This substitution leads to

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} F + ((Rz + S)\theta + a\theta^4 F^2)\frac{\mathrm{d}F}{\mathrm{d}z} + b\theta^4 \frac{\mathrm{d}^3 F}{\mathrm{d}z^3} = 0. \tag{3.11}$$

In view of (3.9), it is obvious that if we choose

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = C_2 \theta^4 \tag{3.12}$$

$F(z)$ satisfies an ordinary differential equation:

$$C_2 F + (C_z z + C_1 + aF^2)\frac{\mathrm{d}F}{\mathrm{d}z} + b\frac{\mathrm{d}^3 F}{\mathrm{d}z^3} = 0. \tag{3.13}$$

Here, C_2 is an arbitrary constant. Equation (3.9) now becomes

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = (C_1 + C_2\sigma)\theta^3. \tag{3.14}$$

The symmetry reduction (3.10) with F, θ and σ as the solutions of (3.13), (3.12) and (3.14), respectively, and z as in (1.3) has been earlier observed by Clarkson and Kruskal [10] and Lakshmanan and Kaliappan [6].

Equation (3.13), on integration, gives the second Painlevé equation:

$$(C_1 + C_2 z)F + aF^3/3 + b\frac{\mathrm{d}^2 F}{\mathrm{d}z^2} = C_3. \tag{3.15}$$

Here, C_3 is a constant of integration. For $C_1 = 0$, $C_3 = I_0 C_2$, (3.15) has an exact solution $F(z) = I_0/z$.

Case 2. $z_{xx} \neq 0$. We now consider the symmetry reductions of (3.1) by making use of the truncated expansion (2.31) with u_0, u_1 and z as arbitrary functions of x and t. We substitute this form into (3.1) and equate the coefficients of same powers of z to obtain the following equations:

$$z^{-4} : -(au_0^3 z_x + 6bu_0 z_x^3) = 0 \tag{3.16}$$

$$z^{-3} : au_0^2 u_{0x} - 2au_0^2 u_1 z_x + b(6u_{0x} z_x^2 + 6u_0 z_x z_{xx}) = 0 \tag{3.17}$$

$$z^{-2} : -z_t u_0 + 2au_0 u_1 u_{0x} - au_0 u_1^2 z_x + au_0^2 u_{1x} + b(-3u_{0xx} z_x - 3u_{0x} z_{xx} - u_0 z_{xxx}) = 0 \tag{3.18}$$

$$z^{-1} : u_{0t} + au_1^2 u_{0x} + 2au_0 u_1 u_{1x} + bu_{0xxx} = 0 \tag{3.19}$$

$$z^0 : u_{1t} + au_1^2 u_{1x} + bu_{1xxx} = 0. \tag{3.20}$$

Equation (3.16) yields the result (3.2), while (3.17) implies that

$$u_1 = -I_1 \frac{z_{xx}}{z_x} \tag{3.21}$$

where $I_1 = I_0/2$. Equations (3.18) may now be simplified to give

$$z_x z_t + aI_1^2 z_{xx}^2 + bz_x z_{xxx} = 0. \tag{3.22}$$

PDEs (3.19) and (3.20) are exactly satisfied.

Since the PDE (3.22) admits an exact solution

$$z(x,t) = 12bC_3 t + C_1 + (C_2^2/3C_3)x + C_2 x^2 + C_3 x^3 \tag{3.23}$$

we have a new two-parameter family of solutions of modified Korteweg–de Vries equation (3.1):

$$\begin{aligned} u(x,t) = I_0 \bigg[& \frac{C_2^4}{9C_3^2} - C_1 C_2 - 12bC_3(C_2 + 3C_3 x)t \\ & + \left(\frac{C_2^3}{C_3} - 3C_1 C_3 \right) x + 4C_2^2 x^2 + 8C_2 C_3 x^3 + 6C_3^2 x^4 \bigg] \\ & \times \bigg\{ \left[12bC_3 t + C_1 + (C_2^2/3C_3)x + C_2 x^2 + C_3 x^3 \right] \\ & \times \left[\frac{C_2^2}{3C_3} + C_2 x + 3C_3 x^2 \right] \bigg\}^{-1}. \end{aligned} \tag{3.24}$$

4. Symmetry reductions of Caudrey–Dodd–Gibbon equation

In this section we use the expansion (1.4) about the singular manifold to discuss the symmetry reductions of the Caudrey–Dodd–Gibbon equation [20]

$$u_t + u_{xxxxx} + 30u_x u_{xx} + 30uu_{xxx} + 180u^2 u_x = 0. \tag{4.1}$$

For now, we have $\alpha = -2$, and

$$u_0 = -z_x^2 \tag{4.2}$$

or

$$u_0 = -2z_x^2. \tag{4.3}$$

As in the previous sections, we may consider either z in the form (1.3) or in a more general form obtained by assuming a finite series solution of (4.1).

Case 1. $z_{xx}=0$. On using z as in (1.3) and t as independent variables in (4.1), we get

$$u_t+(Rz+S)u_z+\theta^5u_{zzzzz}+30\theta^3u_zu_{zz}+30\theta^3uu_{zzz}+180\theta u^2u_z=0 \quad (4.4)$$

where R and S are defined as in (2.4), with θ and σ as arbitrary functions of t. We may now write the solution (1.4) in the form (2.5) with A and B as in (2.6) and (2.7), respectively, and $F(z)$ as

$$F(z)=z^{-2}(F_0+F_1z+F_2z^2+\cdots). \quad (4.5)$$

Here, A_k and B_k, $k\geqslant 0$, are arbitrary functions of t, and F_k are arbitrary constants. When this solution is substituted into (4.4) and the coefficients of like powers of z are equated, we get

$$z^{-7}: -720\theta^5A_0F_0-1080\theta^3A_0^2F_0^2-360\theta A_0^3F_0^3=0 \quad (4.6)$$

$$z^{-6}: (-120\theta^5-1200\theta^3A_0F_0-720\theta A_0^2F_0^2)(A_0F_1+A_1F_0)=0 \quad (4.7)$$

$$z^{-5}: -240\theta^3(A_0F_1+A_1F_0)^2-720\theta^3A_0^2F_0^2$$
$$-720\theta A_0F_0((A_0F_0(A_0F_2+A_1F_1+A_2F_0+B_0)+(A_0F_1+A_1F_0)^2)=0. \quad (4.8)$$

Equations (4.6)–(4.8) yield

$$A_0=\theta^2 \quad (4.9)$$

$$A_1=-F_1\theta^2/F_0 \quad (4.10)$$

$$A_0F_2+A_1F_1+A_2F_0+B_0+\theta^2=0 \quad (4.11)$$

where F_0 is equal to either -1 or -2, and F_1 is an arbitrary constant. Since (4.10) and (4.11) imply that A_1, A_2 and B_0 must be proportional to θ^2, we set $A_{k+1}=B_k=0$, $k\geqslant 0$. The solution may now be read as

$$u(z,t)=\theta^2F(z) \quad (4.12)$$

with θ as an unknown function of t. This solution, when inserted into (4.4), gives

$$2\theta\frac{d\theta}{dt}F+\theta\frac{d\theta}{dt}z\frac{dF}{dz}+\theta^2S\frac{dF}{dz}+\theta^7\left(\frac{d^5F}{dz^5}+30\frac{dF}{dz}\frac{d^2F}{dz^2}+30F\frac{d^3F}{dz^3}+180F^2\frac{dF}{dz}\right)=0. \quad (4.13)$$

It is now obvious that if we have

$$\frac{d\theta}{dt}=-\frac{c_1}{5}\theta^6 \quad (4.14)$$

and

$$S=c_2\theta^5 \quad (4.15)$$

equation (4.13) becomes an ordinary differential equation for $F(z)$:

$$-\frac{c_1}{5}\left(2F+z\frac{dF}{dz}\right)+c_2\frac{dF}{dz}+\frac{d^5F}{dz^5}+30\frac{dF}{dz}\frac{d^2F}{dz^2}+30F\frac{d^3F}{dz^3}+180F^2\frac{dF}{dz}=0. \quad (4.16)$$

If $c_2=0$, this equation has two exact solutions: $F=-1/z^2$; $F=-2/z^2$.

If $c_1\neq 0$, we have

$$\theta(t)=c_3 \qquad \sigma(t)=c_2t+c_4. \quad (4.17)$$

Setting $c_3 = 1$, $c_4 = 0$, we obtain the symmetry reduction

$$u(x,t) = F(z) \qquad z = x + ct. \tag{4.18}$$

If $c_1 \neq 0$, (4.14) and (4.15) give

$$\theta(t) = c_1^{-1/5}(t+c_3)^{-1/5} \tag{4.19}$$

$$\sigma(t) = c_4(t+c_3)^{-1/5} + 5c_2/c_1 \tag{4.20}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. Setting $c_1 = 1$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, we obtain the symmetry reduction

$$u(x,t) = t^{-2/5}F(z) \qquad z = xt^{-1/5}. \tag{4.21}$$

Case 2. $z_{xx} \neq 0$. We now seek the symmetry reductions of the Caudrey–Dodd–Gibbon equation (4.1) by using the truncated expansion

$$u(x,t) = \frac{u_0(x,t)}{z^2} + \frac{u_1(x,t)}{z} \tag{4.22}$$

with $u_0(x,t)$, $u_1(x,t)$ and $z(x,t)$ as arbitrary functions. Substituting this truncated series solution into (4.1), we obtain u_0 as given by (4.2) or (4.3). In the former case, we have

$$u_1(x,t) = z_{xx} \tag{4.23}$$

where z is a solution of the following two PDEs:

$$z_x(z_t + 6z_{xxxxx}) - 15z_{xx}z_{xxxx} + 10z_{xxx}^2 = 0 \tag{4.24}$$

and

$$z_{xt} + z_{xxxxxx} = 0. \tag{4.25}$$

Since

$$z(x,t) = b_0 + b_1 t + b_2 x + \frac{b_3^2}{2b_4}x^2 + b_3x^3 + b_4x^4 + \frac{2b_4^2}{5b_3}x^5 \tag{4.26}$$

satisfies both (4.24) and (4.25), we obtain an exact four-parameter family of solutions:

$$\begin{aligned} u(x,t) = -\Bigg[& b_2^2 - \frac{b_0b_3^2}{b_4} + \left(\frac{b_2b_3^2}{b_4} - 6b_0b_3\right)x + \left(\frac{b_3^4}{2b_4^2} - 12b_0b_4\right)x^2 \\ & + \left(-4b_2b_4 + \frac{2b_3^3}{b_4} - 8\frac{b_0b_4^2}{b_3}\right)x^3 + \left(-\frac{4b_2b_4^2}{b_3} + 4b_3^2\right)x^4 + \frac{28}{5}(b_3b_4x^5 + b_4^2x^6) \\ & + \frac{16b_4^3}{5b_3}x^7 + \frac{4b_4^4}{5b_3^2}x^8 - b_1t\left(\frac{b_3^2}{b_4} + 6b_3x + 12b_4x^2 + \frac{8b_4^2}{b_3}x^3\right)\Bigg] \\ & \times \left[b_0 + b_1t + b_2x + \frac{b_3^2}{2b_4}x^2 + b_3x^3 + b_4x^4 + \frac{2b_4^2}{5b_3}x^5\right]^{-2}. \end{aligned} \tag{4.27}$$

5. Similarity reductions of the Fitzhugh–Nagumo equation

In this section we determine the symmetry reductions of the Fitzhugh–Nagumo equation

$$u_t - u_{xx} + au - (a+1)u^2 + u^3 = 0 \tag{5.1}$$

using the series (1.4); $\alpha = -1$, and

$$u_0 = c_* z_x \tag{5.2}$$

where $c_* = \pm\sqrt{2}$. We now consider the following cases.

Case 1. $z_{xx} = 0$. With z as in (1.3) and t as new independent variables, equation (5.1) becomes

$$u_t + (Rz + S)u_z - \theta^2 u_{zz} + au - (a+1)u^2 + u^3 = 0 \tag{5.3}$$

where R and S are defined as in (2.4). Substituting the solution (2.5) with (2.6)–(2.8) into (5.3) and equating the coefficients of different powers of z to zero, we get

$$z^{-3} : -2\theta^2 A_0 F_0 + A_0^3 F_0^3 = 0 \tag{5.4}$$

$$z^{-2} : A_0 F_0[-S + A_0 F_0\{3(A_0 F_1 + A_1 F_0 + B_0) - (a+1)\}] = 0. \tag{5.5}$$

These equations give

$$A_0 = \theta \qquad F_0 = c_* \tag{5.6}$$

and

$$B_0 = \frac{1}{3}\left(\frac{S}{c_*\theta} + a + 1\right) - \theta F_1 - A_1 F_0 \tag{5.7}$$

where F_1 is an arbitrary constant. Here, we may assume that $A_k = B_{k+1} = 0$, $k \geqslant 1$. The solution may now be read as

$$u(x,t) = \theta F(z) + B_0 + B_1 z. \tag{5.8}$$

On using this in (5.3), we have

$$\begin{aligned}&\left[\frac{\mathrm{d}\theta}{\mathrm{d}t} + (a - 2(a+1)B_0 + 3B_0^2)\theta\right] F + \frac{\mathrm{d}B_0}{\mathrm{d}t} + B_1 S + aB_0 - (a+1)B_0^2 + B_0^3 \\ &\quad + \theta(Rz + S)\frac{\mathrm{d}F}{\mathrm{d}z} + (3B_0 - (a+1))\{z^2 B_1^2 + \theta^2 F^2 + 2z\theta B_1 F\} + B_1^3 z^3 \\ &\quad - \theta^3 \frac{\mathrm{d}^2 F}{\mathrm{d}z^2} + \theta^3 F^3 + 3\theta z^2 B_1^2 F + 3z\theta^2 B_1 F^2 \\ &\quad + \left[\frac{\mathrm{d}B_1}{\mathrm{d}t} + (R + a - 2(a+1)B_0 + 3B_0^2)B_1\right] z = 0.\end{aligned} \tag{5.9}$$

It now easily follows that if (5.9) is to become an ordinary differential equation for $F(z)$, we must have B_1 proportional to θ, and θ must be a constant. Setting

$$B_0 = \frac{a+1}{3} \qquad B_1 = 0 \tag{5.10}$$

$$\theta = 1 \qquad \sigma = c_1 t \tag{5.11}$$

$$F_1 = -c_1/3c_* \tag{5.12}$$

we obtain the travelling wave solution [7].

Case 2. $z_{xx} \neq 0$. When we substitute the solution (1.4) into (5.1) and equate the coefficients of different powers of z to zero, u_0 is as given by (5.2), and

$$z^{-2} : -Su_0 - (a+1)u_0^2 + 3u_0^2 u_1 + 3u_0 z_{xx} = 0 \tag{5.13}$$

$$z^{-1} : u_{0t} + (a - R - 2(a+1)u_1 + 3(u_1^2 + u_0 u_2))u_0 - u_{0xx} = 0 \tag{5.14}$$

$$\begin{aligned}z^0 : u_{1t} &+ (S - 2(a+1)u_0 + 6u_0 u_1)u_2 + au_1 - (a+1)u_1^2 + u_1^3 + 3u_0^2 u_3 \\ &- u_{1xx} + u_{2x} z_x + u_2 z_{xx} = 0.\end{aligned} \tag{5.15}$$

(a) To consider a truncated series solution of type (2.31), we set $u_k = 0$, $k \geqslant 2$. This yields

$$-z_t + 3z_{xx} + (3u_1 - (a+1))c_* z_x = 0 \tag{5.16}$$

$$z_{xt} - z_{xxx} + (a - 2(a+1)u_1 + 3u_1^2)z_x = 0 \tag{5.17}$$

and

$$u_{1t} - u_{1xx} + au_1 - (a+1)u_1^2 + u_1^3 = 0. \tag{5.18}$$

If u_1 is assumed to be a constant, the solutions of (5.18) are $u_1 = 0, 1, a$. With $u_1 = 0$, we have

$$z = c_1 + c_2 \exp[\{c_* x + (1-2a)t\}/2] + c_3 \exp[\{ac_* x + (a^2 - 2a)t\}/2] \tag{5.19}$$

as the solution of (5.16) and (5.17). For positive and negative values of c_*, the solutions were found earlier by Estevez [19].

With $u_1 = 1$ and $u_1 = a$, we obtain the following new solutions:

$$u = \frac{c_2 a \exp[\{c_*(a-1)x + (a^2-1)t\}/2] + c_1}{c_1 + c_2 \exp[\{c_*(a-1)x + (a^2-1)t\}/2] + c_3 \exp[\{-c_* x + (2a-1)t\}/2]} \tag{5.20}$$

and

$$u = \frac{c_2 \exp[\{c_*(1-a)x + (1-a^2)t\}/2] + c_1 a}{c_1 + c_2 \exp(\{c_*(1-a)x + (1-a^2)t\}/2) + c_3 \exp(\{-c_* a x + (2a-a^2)t\}/2)} \tag{5.21}$$

respectively.

(b) Now we seek solutions of type (1.2) when $z_{xx} \neq 0$. Since (5.13) gives

$$u_1 = \frac{a+1}{3} + \left(\frac{S}{3} - z_{xx}\right)\frac{1}{u_0} \tag{5.22}$$

we may consider a symmetry reduction in the form

$$u(x,t) = z_x F(z) + B_0 + B_1 z \tag{5.23}$$

where B_1 is an arbitrary function of t. In view of (2.8), we have $B_0 = u_1 - z_x F_1$, with F_1 as an arbitrary constant. When u as in (5.23) is substituted into (5.1), we get

$$\begin{aligned}
&[z_{xt} - z_{xxx} + (a - 2(a+1)B_0 + 3B_0^2)z_x]F + z_x^3 F^3 + 3B_1^2 z_x z^2 F + B_1 z_x^2 z F^2 + 2z_x^2 B_1 z F^2 \\
&\quad + (3B_0 - (a+1))(z_x^2 F^2 + z^2 B_1^2 + 2z_x B_1 z F) + \frac{\mathrm{d}B_0}{\mathrm{d}t} + (z_t + z_{xx})B_1 + aB_0 \\
&\quad - (a+1)B_0^2 + B_0^3 - z_x^3 \frac{\mathrm{d}^3 F}{\mathrm{d}z^3} + (z_x z_t - 3z_x z_{xx})\frac{\mathrm{d}F}{\mathrm{d}z} \\
&\quad + \left[\frac{\mathrm{d}B_1}{\mathrm{d}t} + B_1\{R + a - 2(a+1)B_0 + 3B_0^2\}\right] z = 0.
\end{aligned} \tag{5.24}$$

It is again obvious that if (5.24) is to become an ordinary differential equation for $F(z)$, B_0 and B_1 must be defined as in (5.10), and z must satisfy

$$z_t = 3z_{xx} \tag{5.25}$$

and

$$z_{xt} - 3z_{xxx} + (a - (a+1)^2/3)z_x = 0 \tag{5.26}$$

where $a = -1, \frac{1}{2}, 2$. For these values of a, we obtain solutions (21)–(22) and (23)–(24) of Nucci and Clarkson.

6. Conclusion

In the present paper we have determined the symmetry reductions of the Burgers' equation, modified Korteweg–de Vries equation, Caudrey–Dodd–Gibbon equation and the Fitzhugh–Nagumo equation, using the singular manifold method. We have found that expansion (1.4) must be used in three ways. First, if the expansion variable z is in the form (1.3), the symmetry reductions of type (1.2) given by the direct method of Clarkson and Kruskal are recovered. The application of the series is indeed simpler to obtain the same results.

Second, the truncation of the series (1.4) at the constant level term yields all the non-classical exact symmetry reductions given by the method of Bluman and Cole. These solutions require that the similarity variable be the solution of certain PDEs. The requirement is derived by the consistency conditions of the series solution. Different forms of the constant level term lead to many symmetry reductions which have not been found heretofore. As a matter of fact, all the solutions of Burgers' equation [13, 14] and the Fitzhugh–Nagumo equation [7] are truncated series solutions. For the former equation, the special values of the parameters in the similarity variable can sometimes allow it to be a function of z as in (1.3). The exact solutions thus obtained always satisfy the consistency criterion of Arrigo *et al* for the equivalence of the methods of Clarkson and Kruskal and Bluman and Cole.

Third, the infinite series (1.4) can be summed exactly to the form (1.2) even when $z_{xx} \neq 0$. The solutions may now be 'N-soliton' type. This case gives rise to a few of 'two-soliton' type solution of Fitzhugh–Nagumo equations. These solutions can also be obtained via the direct method of Clarkson and Kruskal.

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